THE RECURSIVE LOGIT MODEL TUTORIAL

CIRRELT SEMINAR
August 2017

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TUTORIAL GOALS

- The recursive logit model is a random utility model for the choice of path in a network with no restriction on the choice set. It is based on dynamic discrete choice theory.
- This tutorial is focused on theory and practice and aims to
  - describe the problem, the model and its advantages
  - familiarize the reader with the open-source Matlab code

PREREQUISITES

We assume the reader is familiar with discrete choice theory and refer to the textbook of Ben-Akiva and Lerman for insights on this topic.

- Ben-Akiva, M. and Lerman, S. R. Discrete Choice Analysis: Theory and Application to Travel Demand. MIT Press, Cambridge, Massachusetts, 1985.

OUTLINE

- Introduction
- Theory
  - Route choice modeling approaches
  - Recursive logit model formulation
  - Maximum likelihood estimation
- Practice
  - Matlab
  - Code structure
  - Data files
  - A first example

1. Introduction: the problem
2. Theory: the models
3. Practice: making it work

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ROUTE CHOICE MODELING

The problem

- Given an origin and destination in a transport network, **which route** does a traveler choose?
- Travelers do not always choose the shortest path in terms of distance
- Other **attributes** affect the choice through a generalized cost function

MODELING FRAMEWORK

Discrete choice framework

- Analyst **observes path choices** but has **imperfect knowledge** of travelers' generalized cost and perception of network
- Parameters to be **estimated** on such data describe individuals' preferences for attributes
- The estimated models define **choice probability distributions** over alternative paths

OBJECTIVES

A model that can be

- **Consistently estimated** in reasonable time using path choice data collected in real large-scale networks
- used for **accurately predicting** path choices in short computational time (e.g. in a traffic simulation context)

WHY IT IS DIFFICULT

The choice set problem

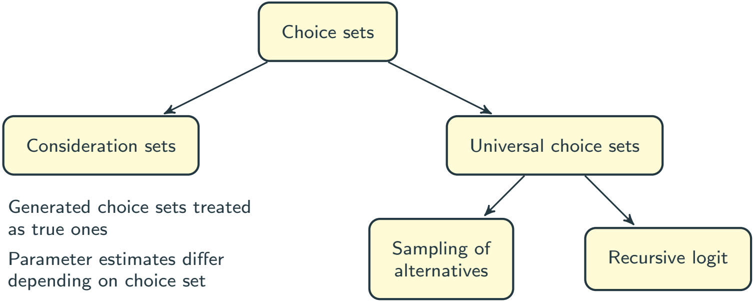
- We don't know **what alternatives** individuals consider and there are infinitely many paths connecting each OD pair

The correlation problem

- Many paths **overlap** in a real network and overlapping paths probably share unobserved attributes

ROUTE CHOICE MODELS

Route choice models can be categorized according to the way they deal with **choice sets**.

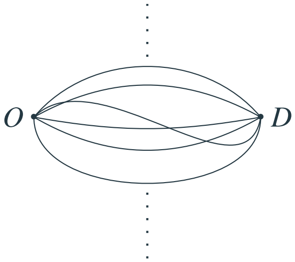


- Generated choice sets treated as true ones
- Parameter estimates differ depending on choice set
- Issues with prediction

- Consistent estimates
- Issues with prediction

- Consistent estimates
- Straightforward for prediction

"CONSIDERATION SETS" APPROACH



- Choice of path modeled as **selection from a discrete set of routes**
- Since the set of feasible routes between O and D cannot be enumerated, the modeler generates a subset of path alternatives.
- The generated choice set is treated as the true one.
- **Problem:** Parameter estimates may significantly vary depending on the choice set definition!

ANOTHER APPROACH

Recursive logit model

- Proposed by Fosgerau, Frejinger and Karlström, (2013). A link-based network route choice model with unrestricted choice set, Transportation Research Part B 56(1):7080.
- Choice of path is formulated as a **sequence of link choices**
- A specific case of **dynamic discrete choice**

DYNAMIC DISCRETE CHOICE

- Horizon T, steps $t = 1, ..., T$
- Individuals make **sequential decisions** over horizon
- A **state** s_t describes all the information relevant for the individual at step t
- An **action** a_t is the decision taken at step t, affecting the value of future state s_{t+1}
- Instantaneous **utility** $u(a_t|s_s)$ of choosing action a_t
- A **Markov transition density function** $F(s_{t+1}|a_t, s_t)$ describes the evolution of future states

DYNAMIC DISCRETE CHOICE

- **Unobserved** state variables ε_t, **observed** state variables x_t
- ε_t is an i.i.d. extreme value type I random variable
- Utility separable in $u(a_t|s_t) = v(a_t|x_t) + \varepsilon_t$
- $v(a_t|x_t)$ parametrized by β to be **estimated**
- Markov transition function becomes $F(\varepsilon_{t+1})F(x_{t+1}|a_t,x_t)$
- Individual chooses action a_t which maximizes instantaneous and future utility
- **Expected maximum utility** at state x_t given by integrated **Bellman's equation**

$$V(x_t) = E_\varepsilon \left[\max_{a_t} \left(v(a_t|x_t) + \varepsilon_t + \int V(x_{t+1}) dF(x_{t+1}|a_t,x_t) \right) \right]$$

RECURSIVE LOGIT MODEL

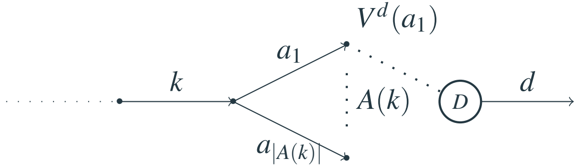
A dynamic discrete choice model for the choice of path

- The network is represented by a graph $G = (A, V)$
- A state $k \in A$ is a **link** in the network
- An action $a \in A(k)$ is an **outgoing link** at the sink node of k
- Infinite horizon but **absorbing link** d with no successor corresponding to destination.
- A path is a sequence of states $k_0, \dots, k_I$ with $k_{i+1} \in A(k_i) \quad \forall i$ and $k_I = d$

ATTRIBUTES

- We denote $x(a|k)$ the attributes of the **link pair** (k,a)
- Parameters β describe **individual preferences** regarding attributes
- $u(a|k)$ is the random utility of link a given current link k
- $u(a|k) = v(a|k) + \mu\varepsilon(a), \varepsilon(a)$ i.i.d EV type I
- $v(a|k) = \beta^T x(a|k)$
- Attributes must be **link-additive** and **deterministic**

LINK CHOICE SITUATION



- Traveler chooses next link a given current state k
- Next state k_{t+1} is **given with certainty** by the action a_t since $k_{t+1} = a_t$
- Traveler chooses action $a \in A(k)$ that maximizes sum of $u(a|k)$ and expected maximum utility to destination $V^d(a)$, denoted the **Value function**

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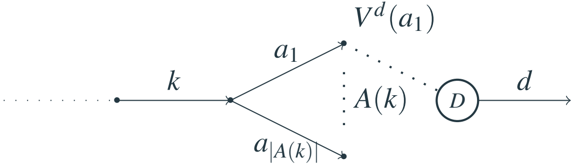
LINK CHOICE PROBABILITIES



- Value function to destination given by Bellman's equation

$$V^d(k) = E_\varepsilon \left[\max_{a \in A(k)} \left\{ v(a|k) + V^d(a) + \mu\varepsilon(a) \right\} \right] \qquad (1)$$

- Link choice probability given by **logit model**

$$P^d(a|k) = \frac{e^{\frac{1}{\mu}v(a|k)+V^d(a)}}{\sum_{a' \in A(k)} e^{\frac{1}{\mu}v(a'|k)+V^d(a')}}. \qquad (2)$$

SOLVING THE VALUE FUNCTION

- The exponential of the Value function in (1) can be rewritten

$$e^{\frac{1}{\mu}V(k)} = \begin{cases} \sum_{a \in A} \delta(a|k) e^{\frac{1}{\mu}(v(a|k)+V(a))} & \text{if } k \in A, \\ 1 & \text{if } k = d. \end{cases}$$

- The Value function is obtained by **solving a system** of linear equations

$$z = Mz + b \Leftrightarrow z = (I - M)^{-1} b$$

where

$$z_k = e^{\frac{1}{\mu}V(k)}$$

$$b_k = 0 \ \forall k \in A, \ b_d = 1$$

$$M_{ka} = \delta(a|k) e^{\frac{1}{\mu}v(a|k)}$$

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PATH CHOICE PROBABILITIES

- Path $\sigma = k_0, \dots, k_I$ where $k_I = d$ with choice probability

$$
\begin{aligned}
P(\sigma) &= \prod_{i=0}^{I-1} P^d(k_{i+1}|k_i) \\
&= \frac{e^{\frac{1}{\mu}\sum_{i=0}^{I-1} v(k_{i+1}|k_i)}}{e^{\frac{1}{\mu}V^d(k_0)}} \\
&= \frac{e^{\frac{1}{\mu}v(\sigma)}}{\sum_{\sigma' \in \mathscr{U}} e^{\frac{1}{\mu}v(\sigma')}}.
\end{aligned}
$$

- The RL model is equivalent to a static multinomial logit model with universal choice set $\mathscr{U}$

WHY IS THE RL BETTER?

Advantages over path-based models

- Avoids generating choice sets of paths both for estimation and prediction
- Parameter estimates are consistent
- Efficient for prediction

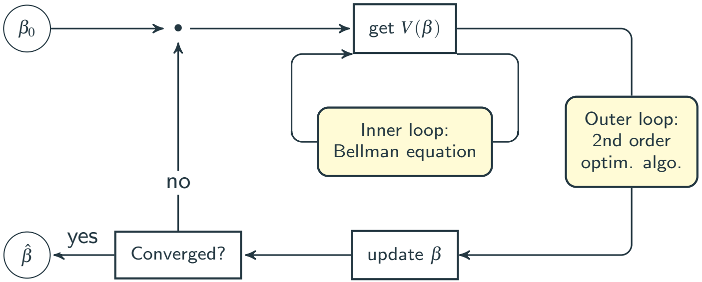
MAXIMUM LIKELIHOOD ESTIMATION

- Data of observed **path choices** $\sigma_n,\ n = 1, ...N$.
- Maximum likelihood estimation problem

$$\max_{\beta} \sum_{n=1}^{N} \ln P(\sigma_n; \beta)$$

- Estimation requires to combine **inner and outer algorithm**, e.g. Nested Fixed Point (NFXP) algorithm
  - Outer algorithm: solves the non-linear optimization problem
  - Inner algorithm: solves the Value functions

MAXIMUM LIKELIHOOD ESTIMATION



PREDICTION

- Depending on applications, it may be useful to
  - Sample a path from the estimated distribution (e.g. applications with scenarios)
  - Predict expected link flows assuming a fixed demand (e.g. traffic assignment applications)

Advantages of RL model

- Allows path sampling **without choice set generation**
- Paths sampled according to the **true estimated probabilities**
- Possibility to fastly compute expected link flows **without repeated path sampling**

PREDICTION

Path sampling

- Sequentially sample arcs $k_0, k_1, \ldots$ until reaching arc d according to estimated link choice probabilities $P^d(a|k)$ $\forall k, a \in A$ in (2)

Expected link flows

- P^d: link choice probabilities $\forall k, a \in A$
- G^d: demand originating at $a \in A$ and ending at d
- F^d: expected flow towards d on $a \in A$ obtained by solving

$$F^d(a) = G^d(a) \sum_{k \in A} P^d(a|k) F(k) \tag{3}$$

LINK SIZE ATTRIBUTE

- Similar to Path Size attribute for path-based models
- Heuristically **corrects utility** of overlapping paths

Computing the LS attribute

- Choose utility with parameters $\tilde{\beta}$
- For each OD pair, compute expected link flow F^{OD} with (3) where G is zero-valued except for $G(O) = 1$.
- Link size attribute is expected link flow

$$LS^{OD} = F^{OD}$$

- Note: the LS attribute is **origin-destination specific!**

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MATLAB

- The implementation of the RL model is in MATLAB
- Matlab documentation reference:
  https://www.mathworks.com/help/matlab/

GET THE CODE

- The Recursive logit code is available on GitHub here: https://github.com/maitien86/RL-Tutorial
- You can clone the following repository

```
$ git clone git://github.com/maitien86/RL-Tutorial
```

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DESCRIPTION

- This tutorial is aimed at users who want to use the code to estimate a path choice model with their own data
- We will go through the **type of input data** needed and the **main functions** of the code
- We will show how the code works on an **illustrative dataset**

QUICK OVERVIEW

- You will need to handle the following files
  - `loadData.m`
    Loads the network data and observations given in the Input folder.
  - `initializeOptStruct.m`
    Tunes estimation algorithm and model parameters.
  - `RLoptimizer.m`
    Begins the maximum likelihood estimation algorithm and returns estimates.

ILLUSTRATIVE DATASET

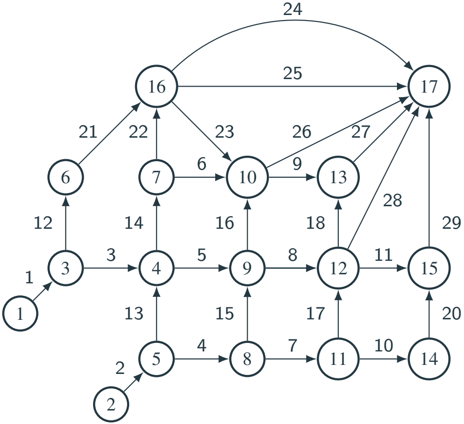

Figure: Example network labeled with link IDs

DATA FILES

- The code requires the following data files to be placed in the Input folder:
  - `LinkAttributes.txt`
    A file containing attributes values for all links
  - `Incidence.txt`
    The matrix representation of the graph G
  - `Observations.txt`
    Link-by-link descriptions of observed itineraries

LINK ATTRIBUTES

- This file describes the attributes of each network link. It consists of several columns containing each an attribute value. The first three columns should indicate
  - The link ID,
  - The ID of the front node,
  - The ID of the end node.
- In practice for real networks such a file can often be obtained from GIS data.

LinkAttributes.txt

```
LinkID,FromNode,ToNode,Length,Trafficsignal
1,1,3,1,0
2,2,5,1,0
3,3,4,1.5,1
4,5,8,1.2,0
5,4,9,1.2,1
```

Figure: Link attributes file for example network

LINK ATTRIBUTES

- The `LinkAttributes` matrix can be read from the link attributes file

Reading the link attributes file

```
file_linkAttributes='./Input/LinkAttributes.txt';
linkAttributes = csvread(file_linkAttributes,1,0);
```

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INCIDENCE MATRIX

- This file describes the incidence matrix of the graph G.
- In practice it can be directly generated from the `LinkAttributes` matrix.

Generate incidence matrix

```
nLinks = length(LinkAttributes(:,1));
A = LinkAttributes(:,3);
B = LinkAttributes(:,2);
Incidence = sparse(nLinks,nLinks);
for i = 1:nLinks
   U = find(B == A(i));
   Incidence(i,U) = 1;
end
```

INCIDENCE MATRIX

- The `Incidence` matrix should also include **dummy links** for each observed destination
- In the illustrative example we have 29 network links and we consider a single destination corresponding to node 17
- An absorbing link (here labeled 30) should be added to the set of network links
- An added column should be added to the `Incidence` matrix, of final size 29×30

INCIDENCE MATRIX

- The `Incidence` matrix can be saved as a text file so it can be easily loaded for future use

Save and load the incidence matrix

```
[i,j,val] = find(Incidence);
data_dump = [i,j,val];
save('IncidenceMatrix.txt','data_dump','-ascii');
```

```
file_incidence='./Input/IncidenceMatrix.txt';
Incidence = spconvert(load(file_incidence));
```

OBSERVATIONS

- This file describes observed trajectories in terms of sequences of link IDs.
- The destination link should be repeated at the beginning of the sequence.

Sample of observations in example network

Obs.	dest.	orig.			links			
1	30	1	3	14	6	26	30	
2	30	1	3	5	8	11	29	30
3	30	1	3	5	8	28	30	
4	30	1	3	5	8	28	30	
5	30	1	12	21	25	30		

OBSERVATIONS

- In practice data processing steps may be required to obtain observed data in the desired format
- Similarly to the `Incidence` matrix, observations should be saved as a text file that can be loaded as a sparse matrix `Obs`

Save and load observation matrix

```
[i,j,val] = find(Obs);
data_dump = [i,j,val];
save('Observations.txt','data_dump','-ascii');
```

```
file_incidence='./Observations.txt';
Obs = spconvert(load(file_observations));
```

SPECIFYING ATTRIBUTES

Attributes are specified in the `loadData.m` file. Since the RL model requires link pairs attributes, this require several steps.

1. extract from the `LinkAttributes` matrix the attribute columns to be included in the model specification,
2. transform link attributes vectors into link pair attributes matrices,
3. store each attribute matrix into the `Atts` ObjArray variable.

EXAMPLE SPECIFICATION

- In the illustrative example, we specify 3 link pair attributes
  - Link length
  - Presence of traffic signal
  - Link constant
- Attributes are defined for link pairs (k,a) and may be independent of state k (e.g. link length of a) or dependent on both states (e.g. turn angle between links k and a)

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SETTING PARAMETERS

- Parameters are set in the `initializeOptStruct.m` file
- The first set of parameters to tune is related to the estimation algorithm.

Op.OptimMethod	Whether to use a line search or trust region method
Op.HessianApprox	Whether to use a BFGS or BHHH Hessian approximation
Op.maxIter	The maximum number of iterations of the algorithm

- The second set consists of model parameters.

Op.n	The number of attributes in the utility specification
Op.LinkSize	A boolean to include or not a Link Size attribute

RUNNING THE ESTIMATION ALGORITHM

- The main file `RLoptimizer.m` starts the model estimation procedure
- The files `loadData.m` and `initializeOptStruct.m` are called within this file
- Details of each iteration are reported
- The value of estimated parameters and standard deviation are displayed at the end and can optionally be saved in the Results folder

EXAMPLE OUTPUT

```
The algorithm stops, due to RELATIVE GRADIENT
The attributes are
[Iteration]: 15
     LL = 2.381999
     x =
         -2.358565e+00
         -3.327645e-01
         2.640459e-02
     norm of step = 0.000150
     radius = 0.000601
     Norm of grad = 0.000002
     Norm of relative gradient = 0.000000
     Number of function evaluation = 20.000000

 Number of function evaluation 20

 Estimated time 5.739890e-01
```

Figure: Output of model estimation for illustrative example